MA 222 - Analysis II: Measure and Integration (JAN-APR, 2016)

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Problem Set 5

1. Let ν be a signed measure on $(X, \Im)$. Show that

$$\nu^+ E = sup\{\nu F : F \subset E; E, F \in \Im\}$$

$$\nu^- E = inf\{\nu F : F \subset E; E, F \in \Im\}$$

2. Let $(X, \Im, \mu)$ be a measure space and f be integrable. Define $\nu E = \int_E f d\mu$. Show that
   (a) ν is a signed measure on $(X, \Im)$ and it is a measure if $f \geq 0$
   (b) $| \nu |$ (E) $= \int_E |f| d\mu$
   (c) Define F(x)$= \int_{-\infty}^{x} f$ is continuous, where $(X, \Im, \mu)$ is the L-measure space in $\mathbb{R}$.
   (d) Find the Hahn decomposition and Jordan decomposition for ν.

3. Let μ, ν be two measures. Show that $\nu \ll \mu$ does not imply $\mu E = 0 \implies \nu E = 0$ by constructing a suitable example.

4. Let μ be a signed measure. Calculate $\frac{d\mu^+}{d\mu}, \frac{d\mu^-}{d\mu}, \frac{d\mu}{d|\mu|}$.

5. (a) Let ν, μ, λ be signed measures and let $\nu \ll \mu, \mu \ll \lambda$. Show that $\nu \ll \lambda$ and $\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda}$.
   (b) Let $\mu \ll \nu$ and $\nu \ll \mu$. Show that $\frac{d\mu}{d\nu} = \frac{1}{\frac{d\nu}{d\mu}}$. Deduce that $\frac{d\mu}{d\nu} \neq 0$ a.e.
   (c) Let f be integrable. Show that

$$\int f d\mu = \int f \frac{d\mu}{d\lambda} \lambda.$$

6. Let ν, μ be finite positive measures and $\nu \ll \mu$. If for any measurable function $g \geq 0$, we have

$$\int g d\mu = \int f g d(\mu + \nu),$$

where $f = \frac{d\nu}{d(\mu+\nu)}$. Prove that $\frac{d\nu}{d\mu} = \frac{f}{1-f}$.

7. Construct a counter example for the continuity theorem that ν is a finite signed measure cannot be dropped. (Recall the theorem: ν is a finite signed measure, μ is a signed measure with that $\nu \ll \mu$. Thus, $\epsilon > 0, \exists \delta > 0$ such that $|\mu|(E) \leq \epsilon \implies |\nu|(E) \leq \delta$).

8. Construct an example that μ is σ-finite cannot be dropped in R-N Theorem.

9. Let ν be any signed measure on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ and μ be the conting measure. Then, show that $\nu \ll \mu$ and compute $\frac{d\nu}{d\mu}$.

10. Let μ and ν be σ-finite measures and $\nu \ll \mu$, then, show that $\frac{d\nu}{d\mu} \neq 0$ a.e. $[\nu]$.